XX and Ising limits in integral formulae for finite temperature correlation functions of the XXZ chain

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Abstract

We consider a multiple integral representation for a one-parameter generating function of the finite temperature S^z-S^z correlation functions of the antiferromagnetic spin-$1/2$ XXZ chain in the XX limit and in the Ising limit. We show how in these limits the multiple integrals reduce to single integrals, thereby reproducing known results.

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1 Introduction

The antiferromagnetic spin-$\frac{1}{2}$ XXZ chain with Hamiltonian

$$H = J \sum_{j=1}^{L} \Big(\sigma^x_{j-1}\sigma^x_j + \sigma^y_{j-1}\sigma^y_j + \Delta(\sigma^z_{j-1}\sigma^z_j - 1) \Big) \tag{1}$$

is a rare exception, even among the models solvable by Bethe ansatz and connected to the Yang-Baxter equation, in that much is known exactly about its correlation functions. All the remarkable progress achieved in the past decade originated in the derivation [13,14] of a so-called multiple integral representation for the density matrix of the model. The next key works were [21], where the results of [13,14] were rederived by means of the algebraic Bethe ansatz and generalized to include a non-zero longitudinal magnetic field, then the article [5], where some of the multiple integrals were actually calculated for the first time, and [6], where an ansatz for the explicit form of a special density matrix element was formulated (for the inhomogeneous and isotropic model).

The ideas developed in these works evolved further in various recent articles. The ansatz of [6] for the inhomogeneous generalization of the correlation functions was related to the q-Knizhnik-Zamolodchikov equation, it was generalized to the general density matrix element and finally proved (see [3,4] and the literature cited there). The technique of [5] was developed further, and a number of concrete examples of short range correlation functions of the XXZ chain was worked out, e.g. [15,16]. Within the Bethe ansatz based approach integral representations for correlation functions other than the density matrix elements were developed [18], the important problem of the summation of density matrix elements into two-point functions was attacked successfully [20], and multiple integral representations for time-dependent correlation functions were suggested [19].

Based on a combination of the Bethe ansatz approach with quantum transfer matrix techniques [11,22,23,26] the authors in collaboration with A. Klümper derived multiple integral representations for finite temperature correlation functions [9] and density matrix elements [8,10].

This article is part of a programme of exploring the properties of finite temperature correlation functions from their multiple integral representations. So far the multiple integrals have proved to be useful for deriving high temperature expansions for correlation functions [28] and have also been evaluated numerically in the simplest non-trivial situation of three lattice sites [7]. Here we shall show how the multiple integrals reduce to single integrals in the XX limit and in the Ising limit. As far as the XX limit is concerned our discussion in many respects parallels that for the zero temperature case [17].

The paper is organized as follows. In the next section we review the multiple integral formula for the one-parameter generating function [12] of the S^z-S^z correlation functions of the XXZ infinite chain, that was obtained in [9] and generalizes a formula in [18] to finite temperature. Then we treat the XX limit and the Ising limit in two separate sections. We close with a short summary.

2 Integral representation of the generating function

Starting point of our considerations in the following two sections will be the multiple integral representation for a one-parameter generating function of the S^z-S^z correlation functions derived in [9]. This one-parameter generating function is the thermal average

$$\Phi(\varphi|m) = \left\langle \exp\{\varphi \sum_{j=1}^{m} e_{j\,2}^{\;2}\} \right\rangle_{T,h}, \tag{2}$$

where $e_2^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and h denotes the value of the longitudinal magnetic field. The function $\Phi(\varphi|m)$ generates a number of interesting correlation functions. We shall consider the two-point function

$$\left\langle S_1^z S_m^z \right\rangle_{T,h} = \tfrac{1}{4}(2D_m^2 \partial_\varphi^2 - 4D_m \partial_\varphi + 1)\Phi(\varphi|m)\Big|_{\varphi=0} \tag{3}$$

and the so-called emptiness formation probability

$$P_\uparrow(m) = \lim_{\varphi\to-\infty} \Phi(\varphi|m) = \left\langle \prod_{j=1}^{m} e_{j\,1}^{\;1} \right\rangle_{T,h} \tag{4}$$

which is the probability to find a ferromagnetic string of length m in the equilibrium state with temperature T and magnetic field h, and is a special matrix element of the density matrix. D_m in (3) denotes the 'lattice derivative' defined on any complex sequence $(z_n)_{n\in\mathbb{N}}$ by $D_m z_m = z_m - z_{m-1}$, and e_1^1 in (4) is the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

We found in our previous works [9, 10] that the same auxiliary function $\mathfrak{a}$ that determines the free energy of the model also contains all the information about the state of thermal equilibrium that is necessary to express the correlation functions as multiple integrals. As will become particularly clear in the XX limit, the function $1/(1+\mathfrak{a})$ generalizes the Fermi function to the interacting case. Since we expect the Fermi function to appear naturally in all expressions for correlation functions, we think that the formulation of the thermodynamics of integrable models as developed in [22, 23] is particularly useful in the present context.

The auxiliary function is the unique solution of the non-linear integral equation

$$\ln \mathfrak{a}(\lambda) = -\frac{h}{T} - \frac{2J\,\mathrm{sh}^2(\eta)}{T\,\mathrm{sh}(\lambda)\,\mathrm{sh}(\lambda+\eta)} - \int_C \frac{d\omega}{2\pi i} \frac{\mathrm{sh}(2\eta)\ln(1+\mathfrak{a}(\omega))}{\mathrm{sh}(\lambda-\omega+\eta)\,\mathrm{sh}(\lambda-\omega-\eta)}\,. \tag{5}$$

Here the integration contour $\mathcal{C}$ is a rectangle in the complex plane. We have sketched it in figure 1. Like the inhomogeneity and the kernel in (5) it depends on the parameter η, which is the only essential parameter of the model and parametrizes the anisotropy Δ in (1) as $\Delta = \mathrm{ch}(\eta)$. Knowing the auxiliary function on $\mathcal{C}$ we can calculate the free energy per lattice site according to the formula [9]

$$f(T,h) = -\frac{h}{2} - T\int_C \frac{d\omega}{2\pi i} \frac{\mathrm{sh}(\eta)\ln(1+\mathfrak{a}(\omega))}{\mathrm{sh}(\omega)\,\mathrm{sh}(\omega+\eta)}\,. \tag{6}$$

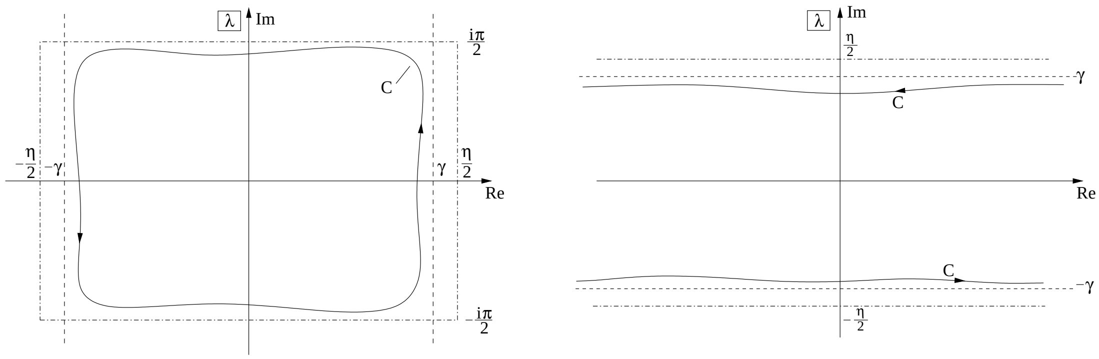

Figure 1: The canonical contour $\mathcal{C}$ in the off-critical regime $\Delta > 1$ (left panel) and in the critical regime $|\Delta| < 1$ (right panel). For $\Delta > 1$ the contour is a rectangle with sides at $\pm i\frac{\pi}{2}$ and $\pm\gamma$, where γ is slightly smaller than $\frac{\eta}{2}$. For $|\Delta| < 1$ the contour surrounds the real axis at a distance $|\gamma|$ slightly smaller than $\frac{|\eta|}{2}$ if $\Delta > 0$ and slightly smaller than $\frac{\pi}{2} - \frac{|\eta|}{2}$ if $\Delta < 0$.

We can now state the main result of [9]. The generating function $\Phi(\varphi|m)$ of the correlation functions of the XXZ chain in the thermodynamic limit, as defined in (2), has the following integral representation:

$$\Phi(\varphi|m) = \sum_{n=0}^{m} \frac{e^{(m-n)\varphi}}{(n!)^2} \left[\prod_{j=1}^{n} \int_{\Gamma} \frac{d\zeta_j}{2\pi i} \left(\frac{\operatorname{sh}(\zeta_j - \eta)}{\operatorname{sh}(\zeta_j)} \right)^m \int_{\mathcal{C}} \frac{d\omega_j}{2\pi i(1+\mathfrak{a}(\omega_j))} \left(\frac{\operatorname{sh}(\omega_j)}{\operatorname{sh}(\omega_j - \eta)} \right)^m \right] \left[\prod_{j,k=1}^{n} \frac{\operatorname{sh}(\omega_j - \zeta_k - \eta)}{\operatorname{sh}(\zeta_j - \zeta_k - \eta)} \right] \det_n M(\omega_j, \zeta_k) \, \det_n G(\omega_j, \zeta_k) \,, \quad (7)$$

where $\mathfrak{a}(\lambda)$ is the auxiliary function (5), where

$$M(\omega_j, \zeta_k) = t(\zeta_k, \omega_j) \prod_{\ell=1}^{n} \frac{\operatorname{sh}(\omega_j - \zeta_\ell - \eta)}{\operatorname{sh}(\omega_j - \omega_\ell - \eta)} + t(\omega_j, \zeta_k) e^{\varphi} \prod_{\ell=1}^{n} \frac{\operatorname{sh}(\omega_j - \zeta_\ell + \eta)}{\operatorname{sh}(\omega_j - \omega_\ell + \eta)} \,, \quad (8)$$

$$t(\zeta, \omega) = \frac{\operatorname{sh}(\eta)}{\operatorname{sh}(\zeta - \omega) \operatorname{sh}(\zeta - \omega + \eta)} \,, \quad (9)$$

and where the function $G(\lambda, \zeta)$ is the solution of the linear integral equation

$$G(\lambda, \zeta) = t(\zeta, \lambda) + \int_{\mathcal{C}} \frac{d\omega}{2\pi i(1+\mathfrak{a}(\omega))} \frac{\operatorname{sh}(2\eta) G(\omega, \zeta)}{\operatorname{sh}(\lambda - \omega + \eta) \operatorname{sh}(\lambda - \omega - \eta)} \,. \quad (10)$$

The contour $\mathcal{C}$ in the latter equation is the same as in the non-linear integral equation (5) (see figure 1). Γ is any simple closed contour which lies inside $\mathcal{C}$ and encircles the origin counterclockwise.

An alternative expression for $\Phi(\varphi|m)$ which involves the function $\overline{\mathfrak{a}} = 1/\mathfrak{a}$ and which was also obtained in [9] is

$$\Phi(\varphi|m) = \sum_{n=0}^{m} \frac{(-1)^n}{(n!)^2} \left[\prod_{j=1}^{n} \int_\Gamma \frac{d\zeta_j}{2\pi\mathrm{i}} \left(\frac{\mathrm{sh}(\zeta_j+\eta)}{\mathrm{sh}(\zeta_j)} \right)^m \int_C \frac{d\omega_j}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_j))} \left(\frac{\mathrm{sh}(\omega_j)}{\mathrm{sh}(\omega_j+\eta)} \right)^m \right] \left[\prod_{j,k=1}^{n} \frac{\mathrm{sh}(\omega_j-\zeta_k+\eta)}{\mathrm{sh}(\zeta_j-\zeta_k+\eta)} \right] \det{}_n M(\omega_j,\zeta_k) \, \det{}_n G(\omega_j,\zeta_k) \, . \quad (11)$$

For the emptiness formation probability (4) we obtained several alternative expressions [8, 9]. In the present context it will turn out to be convenient to work with the inhomogeneous, symmetric version

$$P_\uparrow(m|\{\xi\}) = \frac{1}{m!} \left[\prod_{j=1}^{m} \int_C \frac{d\omega_j}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))} \right] \left[\prod_{j,k=1}^{m} \frac{\mathrm{sh}(\omega_j-\xi_k)\,\mathrm{sh}(\omega_j-\xi_k-\eta)}{\mathrm{sh}(\omega_j-\omega_k-\eta)} \right] \frac{\det_m t(\xi_k,\omega_j) \det_m G(\omega_j,\xi_k)}{\prod_{\substack{j,k=1 \\ j\neq k}}^{m} \mathrm{sh}(\xi_j-\xi_k)} \, , \quad (12)$$

which follows from equation (111) of [9] in the limit $\varphi \to -\infty$, and to perform the limit $\xi_j \to 0$, $j = 1,\dots,m$, at a later stage.

Formulae for the magnetization are the simplest realizations of the above integral representations, e.g.

$$m(T,h) = \langle S^z_j \rangle_{T,h} = \lim_{\xi\to 0} P_\uparrow(1,\xi) - \tfrac{1}{2} = \tfrac{1}{2}\Phi(\mathrm{i}\pi|1) \, . \quad (13)$$

Using (7), (11) in the last expression on the right hand side we obtain [9]

$$m(T,h) = -\frac{1}{2} - \int_C \frac{d\omega}{2\pi\mathrm{i}} \frac{G(\omega,0)}{1+\mathfrak{a}(\omega)} = \frac{1}{2} + \int_C \frac{d\omega}{2\pi\mathrm{i}} \frac{G(\omega,0)}{1+\overline{\mathfrak{a}}(\omega)} \, . \quad (14)$$

In the following sections we shall consider the limits $\eta \to \mathrm{i}\pi/2$ and $\eta \to \infty$ with $J\Delta = c$ fixed. In the former case Δ goes to zero and the Hamiltonian (1) becomes the XX Hamiltonian. This limit is also called the free Fermion limit. The latter limit is called the Ising limit, since in this limit the Hamiltonian (1) turns into the Hamiltonian of the one-dimensional Ising model.

3 The XX limit

In the XX limit $\eta = \mathrm{i}\pi/2$, $\Delta = \mathrm{ch}(\eta) = 0$ we have

$$\mathrm{sh}(\lambda \pm \eta) = \pm\mathrm{i}\,\mathrm{ch}(\lambda) \, , \quad (15)$$

and the equations (5)-(12) simplify considerably. In particular, the kernel of the integral equations (5) and (10) vanishes identically, and the auxiliary function $\mathfrak{a}$ and the density function G are given by the corresponding inhomogeneities,

$$\mathfrak{a}(\lambda) = \exp\Big\{-\frac{1}{T}\Big(h + \frac{4\mathrm{i}J}{\mathrm{sh}(2\lambda)}\Big)\Big\}, \tag{16}$$

$$G(\lambda,\zeta) = -\frac{2}{\mathrm{sh}(2(\lambda-\zeta))}. \tag{17}$$

Note that the function $G(\lambda,\zeta)$ does not depend on the temperature and therefore agrees with its zero temperature limit. For this fact the following calculations bear many similarities with the zero temperature case studied in [17].

Magnetization

We wish to transform all integrals over $\mathcal{C}$ into integrals over the 'Brillouin zone' $[-\pi,\pi]$. To begin with a simple example, we consider the equation (14) for the magnetization. Inserting (16) and (17) we obtain

$$m(T,h) = \frac{1}{2} - \int_{\mathcal{C}} \frac{d\omega}{\pi\mathrm{i}\,\mathrm{sh}(2\omega)} \frac{1}{1+\exp\big(\frac{1}{T}(h+\frac{4\mathrm{i}J}{\mathrm{sh}(2\omega)})\big)}. \tag{18}$$

The contour of integration $\mathcal{C}$ consists of a part $\mathcal{C}_+$ above the real axis and a part $\mathcal{C}_-$ below the real axis (see figure 1). Since the integrand in (18) is analytic in the strips $-\pi/2 < \mathrm{Im}\,\omega < 0$ and $0 < \mathrm{Im}\,\omega < \pi/2$ we may deform $\mathcal{C}_\pm$ into contours parallel to the real axis passing through the points $\pm\mathrm{i}\pi/4$. Then the parameterization

$$\mathcal{C}_+ : \big[\tfrac{\pi}{2},\pi\big] \cup \big[-\pi,-\tfrac{\pi}{2}\big] \to \mathbb{C}, \qquad \omega(p) = \frac{1}{2}\,\mathrm{arth}(\sin(p)) + \frac{\mathrm{i}\pi}{4}, \tag{19a}$$

$$\mathcal{C}_- : \big[-\tfrac{\pi}{2},\tfrac{\pi}{2}\big] \to \mathbb{C}, \qquad \omega(p) = \frac{1}{2}\,\mathrm{arth}(\sin(p)) - \frac{\mathrm{i}\pi}{4} \tag{19b}$$

of $\mathcal{C}$ is uniform, and for every point ω on $\mathcal{C}$ there is a unique $p \in [-\pi,\pi]$ such that

$$\mathrm{sh}(2\omega) = \frac{1}{\mathrm{i}\cos(p)}, \quad \mathrm{ch}(2\omega) = -\mathrm{i}\,\mathrm{tg}\,(p) \tag{20}$$

It follows the familiar expression

$$m(T,h) = \frac{1}{2} - \int_{-\pi}^{\pi} \frac{dp}{2\pi} \frac{1}{1+\exp\big(\frac{h-4J\cos p}{T}\big)}. \tag{21}$$

Emptiness formation probability

The formula for the emptiness formation probability is obtained in a similar way. We first insert (15)-(17) into (12) and use the product formula for the Cauchy determinant. Then

$$P_\uparrow(m|\{\xi\}) = \frac{1}{m!}\left[\prod_{j=1}^m \int_C \frac{d\omega_j}{2\pi \mathrm{i}(1+\mathfrak{a}(\omega_j))}\right] \frac{\prod_{\substack{j,k=1\\ j\neq k}} \mathrm{sh}(\omega_j-\omega_k)\,\mathrm{ch}(\xi_j-\xi_k)}{\prod_{j,k=1}^m \mathrm{sh}(\omega_j-\xi_k)\,\mathrm{ch}(\omega_j-\xi_k)}\,, \tag{22}$$

and the homogeneous limit $\xi_j \to 0$ is obvious,

$$P_\uparrow(m) = \frac{(-1)^{\frac{m(m-1)}{2}}}{m!}\left[\prod_{j=1}^m \int_C \frac{d\omega_j}{\pi \mathrm{i}(1+\mathfrak{a}(\omega_j))}\right] \frac{\prod_{1\le j<k\le m} 4\,\mathrm{sh}^2(\omega_j-\omega_k)}{\prod_{j=1}^m \mathrm{sh}^m(2\omega_j)}\,. \tag{23}$$

Again the contour and the integrand are uniformly parameterized by (19), in particular, using (20),

$$\mathrm{sh}^2(\omega_j-\omega_k) = \frac{1}{2}\mathrm{ch}(2\omega_j)\,\mathrm{ch}(2\omega_k) - \frac{1}{2}\mathrm{sh}(2\omega_j)\,\mathrm{sh}(2\omega_k) - \frac{1}{2} = \frac{\sin^2\left(\frac{p_j-p_k}{2}\right)}{\cos(p_j)\cos(p_k)} \tag{24}$$

for *all* $p_j, p_k \in [-\pi.\pi]$. Inserting (19), (20) and (24) into (23) we end up with

$$P_\uparrow(m) = \frac{1}{m!}\left[\prod_{j=1}^m \int_{-\pi}^{\pi} \frac{dp_j}{2\pi} \frac{1}{1+\exp\left(\frac{4J\cos p_j - h}{T}\right)}\right] \prod_{1\le j<k\le m} 4\sin^2\left(\frac{p_j-p_k}{2}\right). \tag{25}$$

This can be further simplified, if we take into account that $\prod_{1\le j<k\le m} 4\sin^2\left(\frac{p_j-p_k}{2}\right)$ can be expressed as a product of two van-der-Monde determinants. Then

$$P_\uparrow(m) = \det{}_m\left[\int_{-\pi}^{\pi} \frac{dp}{2\pi} \frac{\mathrm{e}^{\mathrm{i}(j-k)p}}{1+\exp\left(\frac{4J\cos p - h}{T}\right)}\right]_{j,k=1,\dots,m} \tag{26}$$

which is the known result, first obtained in [25]. For a careful discussion of the large distance asymptotics of $P_\uparrow(m)$ at zero and non-zero temperature see also [25].

Two-point function

For the calculation of the two-point function $\left\langle S_1^z S_m^z \right\rangle_{T,h}$ we use the integral representation (11). Inserting (15)-(17) and applying the product formula for the Cauchy determinant twice we find

$$\Phi(\varphi|m) =$$
$$\sum_{n=0}^m \frac{(\mathrm{e}^\varphi-1)^n}{(n!)^2}\left[\prod_{j=1}^n \int_\Gamma \frac{d\zeta_j}{2\pi\mathrm{i}} \int_C \frac{d\omega_j}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_j))}\left(\frac{\mathrm{th}(\omega_j)}{\mathrm{th}(\zeta_j)}\right)^m\right]\left[\det{}_n \frac{1}{\mathrm{sh}(\omega_j-\zeta_k)}\right]^2, \tag{27}$$

where $\overline{\mathfrak{a}} = 1/\mathfrak{a}$ is given explicitly in (16). The right hand side has a rather similar structure as in the zero temperature limit, and, at first instance, the same tricks apply (cf. [17]). Since the parameter φ enters (27) in the combination $(\mathrm{e}^{\varphi} - 1)^m$, only the three lowest order terms in the sum remain after applying the differential-difference operator at the right hand side of (3), and

$$\big\langle S_1^z S_m^z \big\rangle_{T,h} = \tfrac{1}{2} D_m^2 (I_1 + I_2) - D_m I_1 + \tfrac{1}{4}\,, \tag{28}$$

where

$$I_1 = \int_\Gamma \frac{d\zeta}{2\pi \mathrm{i}} \int_C \frac{d\omega}{2\pi \mathrm{i}(1+\overline{\mathfrak{a}}(\omega))} \left(\frac{\mathrm{th}(\omega)}{\mathrm{th}(\zeta)}\right)^m \frac{1}{\mathrm{sh}^2(\omega-\zeta)}\,, \tag{29a}$$

$$I_2 = \frac{1}{2}\left[\prod_{j=1}^2 \int_\Gamma \frac{d\zeta_j}{2\pi \mathrm{i}} \int_C \frac{d\omega_j}{2\pi \mathrm{i}(1+\overline{\mathfrak{a}}(\omega_j))} \left(\frac{\mathrm{th}(\omega_j)}{\mathrm{th}(\zeta_j)}\right)^m\right] \left[\det{}_2 \frac{1}{\mathrm{sh}(\omega_j - \zeta_k)}\right]^2. \tag{29b}$$

The Γ-integrals in (29) can be calculated by means of residue calculus. For this purpose we deform the contour Γ as sketched in figure 2. Then

$$\begin{aligned}\int_\Gamma \frac{d\zeta}{2\pi \mathrm{i}} \left(\frac{\mathrm{th}(\omega)}{\mathrm{th}(\zeta)}\right)^m \frac{1}{\mathrm{sh}^2(\omega-\zeta)} &= \int_{\tilde\Gamma} \frac{d\zeta}{2\pi \mathrm{i}} \left(\frac{\mathrm{th}(\omega)}{\mathrm{th}(\zeta)}\right)^m \frac{1}{\mathrm{sh}^2(\omega-\zeta)} \\ &\quad - \mathrm{res}\left\{\left(\frac{\mathrm{th}(\omega)}{\mathrm{th}(\zeta)}\right)^m \frac{1}{\mathrm{sh}^2(\omega-\zeta)}\right\}_{\zeta=\omega} = \frac{2m}{\mathrm{sh}(2\omega)}\,,\end{aligned} \tag{30}$$

where the integral over $\tilde\Gamma$ is zero, because the integrand is periodic with period $\mathrm{i}\pi$ and vanishes asymptotically for $\mathrm{Re}\,\zeta \to \pm\infty$. Inserting (30) into (29a) and comparing with (18) we conclude that

$$I_1 = m\big[\tfrac{1}{2} - m(T,h)\big] = m\big[\tfrac{1}{2} - \big\langle S_j^z \big\rangle_{T,h}\big]\,. \tag{31}$$

Similarly

$$\begin{aligned} I_2 &= I_1^2 - \left[\prod_{j=1}^2 \int_C \frac{d\omega_j\, \mathrm{th}^m(\omega_j)}{2\pi \mathrm{i}(1+\overline{\mathfrak{a}}(\omega_j))}\right]\left[\int_\Gamma \frac{d\zeta}{2\pi \mathrm{i}} \frac{1}{\mathrm{th}^m(\zeta)\,\mathrm{sh}(\zeta-\omega_1)\,\mathrm{sh}(\zeta-\omega_2)}\right]^2 \\ &= I_1^2 - \left[\prod_{j=1}^2 \int_C \frac{d\omega_j}{2\pi \mathrm{i}(1+\overline{\mathfrak{a}}(\omega_j))}\right]\left[\frac{1}{\mathrm{sh}(\omega_1-\omega_2)}\left(\left[\frac{\mathrm{th}(\omega_1)}{\mathrm{th}(\omega_2)}\right]^{\frac{m}{2}} - \left[\frac{\mathrm{th}(\omega_2)}{\mathrm{th}(\omega_1)}\right]^{\frac{m}{2}}\right)\right]^2. \end{aligned} \tag{32}$$

Inserting the above expressions for I_1 and I_2 into (28) and applying the difference operators we arrive at

$$\begin{aligned}&\big\langle S_1^z S_{m+1}^z \big\rangle_{T,h} - \big\langle S_1^z \big\rangle_{T,h} \big\langle S_{m+1}^z \big\rangle_{T,h} = \\ &\qquad \left[\int_C \frac{d\omega}{2\pi \mathrm{i}(1+\overline{\mathfrak{a}}(\omega))} \frac{\partial_\omega \mathrm{th}^m(\omega)}{m}\right]\left[\int_C \frac{d\omega}{2\pi \mathrm{i}(1+\overline{\mathfrak{a}}(\omega))} \frac{\partial_\omega \mathrm{cth}^m(\omega)}{m}\right].\end{aligned} \tag{33}$$

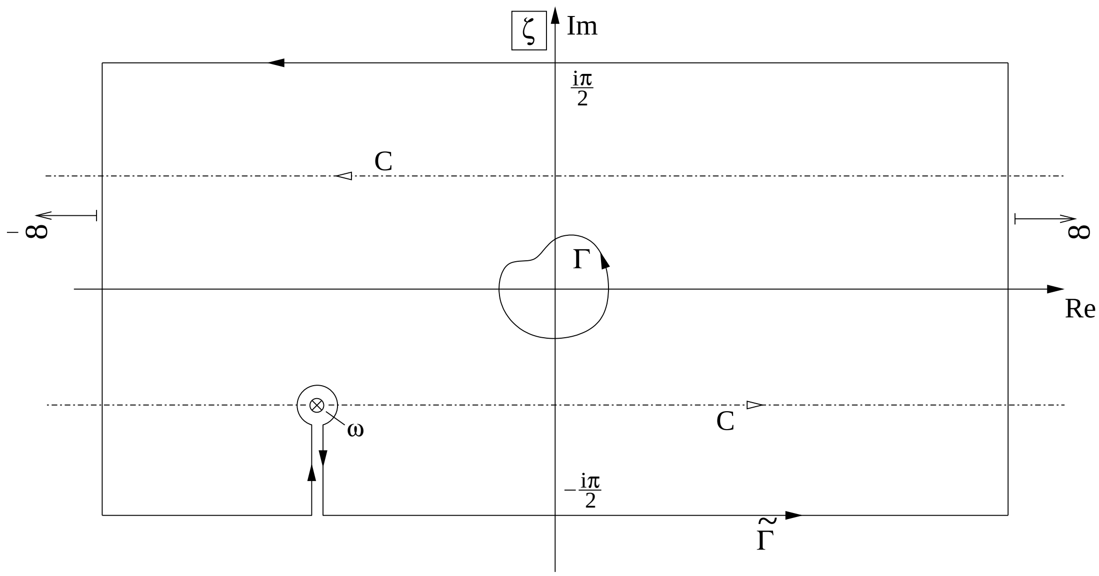

Figure 2: In order to calculate the integrals over ζ in I_1 and I_2 the contour Γ is deformed into $\tilde{\Gamma}$ plus a small circle surrounding the pole at ω.

This is already a rather simple expression involving only products of single integrals. We wrote it in a form that is particularly convenient for transforming it into an integral over the Brillouin zone: Since (see (20))

$$\mathrm{th}(\omega) = \frac{1}{\mathrm{cth}(\omega)} = \frac{\mathrm{ch}(2\omega) - 1}{\mathrm{sh}(2\omega)} = -\mathrm{i}e^{\mathrm{i}p}\,, \tag{34}$$

we see at once that

$$\langle S_1^z S_{m+1}^z \rangle_{T,h} - \langle S_1^z \rangle_{T,h} \langle S_{m+1}^z \rangle_{T,h} = -\left[\int_{-\pi}^{\pi} \frac{dp}{2\pi} \frac{\cos(mp)}{1 + \exp\left(\frac{h - 4J\cos p}{T}\right)} \right]^2 \tag{35}$$

for all $m \in \mathbb{N}$.

Let us finally consider the zero temperature limit. If h is larger than the critical field $4J$, all spins are ferromagnetically aligned at zero temperature and the correlation function (35) vanishes. For $|h| < 4J$ the integral (35) in the zero temperature limit is restricted to the interval between two Fermi points $\pm \arccos\left(\frac{h}{4J}\right)$, where the exponential function in the integrand vanishes, and

$$\langle S_1^z S_{m+1}^z \rangle_{T,h} - \langle S_1^z \rangle_{T,h} \langle S_{m+1}^z \rangle_{T,h} = \frac{T_{2m}\left(\frac{h}{4J}\right) - 1}{2\pi^2 m^2}\,, \tag{36}$$

with $T_j(x)$ being the jth Chebychev polynomial. Equations (35) and (36) are equivalent to formulae first obtained in [24] (for $h = 0$) and [1]. The asymptotics of the two-point functions was studied in [1].

4 The Ising limit

The XXZ Hamiltonian (1) turns into the Hamiltonian of the one-dimensional Ising model if we set $J = \frac{c}{\Delta} = \frac{c}{\mathrm{ch}(\eta)}$, $c > 0$, and take the limit $\eta \to \infty$. In this section we shall reproduce the known formula for the emptiness formation probability in the Ising limit [27] from the multiple integral representation (12). As a prerequisite we need to calculate the auxiliary function $\mathfrak{a}(\lambda)$ and the inhomogeneous density function $G(\lambda,\zeta)$ from their integral equations (5) and (10). We recall that, in the off-critical regime, the integral equations determine the functions $\mathfrak{a}(\lambda)$ and $G(\lambda,\zeta)$ inside the strip $|\mathrm{Im}\,\lambda| \le \frac{\pi}{2}$, $|\mathrm{Re}\,\lambda| \le (1-\delta)\frac{\eta}{2}$ for any fixed δ with $0 < \delta < 1$. Using that

$$\lim_{\eta\to\infty} \mathrm{cth}(\lambda \pm \eta) = \pm 1\,, \tag{37}$$

uniformly for all λ with $|\mathrm{Re}\,\lambda| \le (1-\delta)\eta$, and the elementary formulae

$$\frac{\mathrm{sh}(\eta)}{\mathrm{sh}(\lambda)\,\mathrm{sh}(\lambda+\eta)} = \mathrm{cth}(\lambda) - \mathrm{cth}(\lambda+\eta)\,, \tag{38}$$

$$\frac{\mathrm{sh}(2\eta)}{\mathrm{sh}(\lambda-\eta)\,\mathrm{sh}(\lambda+\eta)} = \mathrm{cth}(\lambda-\eta) - \mathrm{cth}(\lambda+\eta)\,, \tag{39}$$

we see that the integral equations (5), (10) turn into

$$\ln \mathfrak{a}(\lambda) = -\frac{h}{T} - \frac{2c}{T}\big(\mathrm{cth}(\lambda) - 1\big) + \int_C \frac{d\omega}{\pi \mathrm{i}} \ln(1+\mathfrak{a}(\omega))\,, \tag{40}$$

$$G(\lambda,\zeta) = \mathrm{cth}(\zeta-\lambda) - 1 - \int_C \frac{d\omega}{\pi \mathrm{i}} \frac{G(\omega,\zeta)}{1+\mathfrak{a}(\omega)}\,. \tag{41}$$

These equations are now valid for all $\lambda \in \mathbb{C}$ with $|\mathrm{Im}\,\lambda| \le \frac{\pi}{2}$. In particular, we may choose for λ any point on the left or right edge of the contour $\mathcal{C}$ (see figure 1) and then shift the left edge to $-\infty$ and the right edge to $+\infty$. This way we obtain from both integral equations, (40) and (41), two algebraic equations that determine the limits $\mathfrak{a}_\pm = \lim_{|\mathrm{Re}\,\lambda|\to\pm\infty} \mathfrak{a}(\lambda)$ and $G_\pm = \lim_{|\mathrm{Re}\,\lambda|\to\pm\infty} G(\lambda,\zeta)$, respectively. The equations for the limits of the auxiliary functions read

$$\mathfrak{a}_\pm = K \mathrm{e}^{-\frac{h}{T} \mp \frac{2c}{T}}\,, \quad K = \frac{1+\mathfrak{a}_+}{1+\mathfrak{a}_-}\,\mathrm{e}^{\frac{2c}{T}}\,. \tag{42}$$

This is easily turned into a quadratic equation for K whose roots are

$$K = \mathrm{e}^{-\frac{2c}{T} + \frac{h}{2T}} \left(-\mathrm{sh}\Big(\frac{h}{2T}\Big) \pm \sqrt{\mathrm{sh}^2\Big(\frac{h}{2T}\Big) + \mathrm{e}^{\frac{4c}{T}}} \right). \tag{43}$$

Shifting the left and right integration contours to $\mp\infty$ the constant K appears in (40), and we end up with

$$\mathfrak{a}(\lambda) = K \mathrm{e}^{-\frac{h}{T} - \frac{2c}{T}\mathrm{cth}(\lambda)}\,. \tag{44}$$

The sign in (43) is fixed to plus by taking into account that $\mathfrak{a}(\lambda)$ in the high temperature limit is identically equal to 1 for arbitrary value of the anisotropy parameter Δ (see [9]).

The function $G(\lambda,\zeta)$ is obtained in a similar way. The limits $G_\pm$ satisfy

$$G_+ = -2 - \frac{G_+}{1+\mathfrak{a}_+} + \frac{G_-}{1+\mathfrak{a}_-}, \quad G_- = -\frac{G_+}{1+\mathfrak{a}_+} + \frac{G_-}{1+\mathfrak{a}_-} \tag{45}$$

which follows from (41). We infer that $G_\pm$ are independent of ζ and that $G_- - G_+ = 2$. Hence we can introduce a new function, say $\langle\sigma\rangle$, such that

$$G_\pm = \langle\sigma\rangle \mp 1\,. \tag{46}$$

Inserting the latter definition into (45), using (42), (43) and solving for $\langle\sigma\rangle$, we obtain the explicit expression

$$\langle\sigma\rangle = \frac{\mathrm{sh}\bigl(\frac{h}{2T}\bigr)}{\sqrt{\mathrm{sh}^2\bigl(\frac{h}{2T}\bigr) + \mathrm{e}^{\frac{4c}{T}}}}\,. \tag{47}$$

By shifting the integration contour to $\mp\infty$ for fixed λ in (41) and using (45), (46) we eventually arrive at

$$G(\lambda,\zeta) = \mathrm{cth}(\zeta-\lambda) + \langle\sigma\rangle\,. \tag{48}$$

Having determined the auxiliary function $\mathfrak{a}(\lambda)$ and the function $G(\lambda,\zeta)$ we can now calculate the magnetization as a first simple application. Shifting the integration contour to $\mp\infty$ in the first equation (14) and making use of (45) and (46) we obtain at once

$$m(T,h) = \frac{\langle\sigma\rangle}{2}\,, \tag{49}$$

i.e. $\langle\sigma\rangle = \langle\sigma_j^z\rangle_{T,h}$ which explains our notation. Not totally unexpected our formulae (49) and (47) are in agreement with the well-known result [2].

Emptiness formation probability

Starting point for the calculation of the emptiness formation probability is again (12). After (i) taking into account the simplifications due to (37)-(39), (ii) using the formula

$$\det_m \frac{a x_j + b s_k}{x_j - s_k} = (a+b)^{m-1}\left[a\prod_{k=1}^m x_k + b\prod_{k=1}^m s_k\right]\frac{\prod_{1\le j<k\le m}(x_j-x_k)(s_k-s_j)}{\prod_{j,k=1}^m (x_j - s_k)} \tag{50}$$

in order to simplify $\det_m G(\omega_j,\xi_k)$, and (iii) setting $s_k = \mathrm{e}^{-2\xi_k}$ and changing the integration variables to $x_j = \mathrm{e}^{-2\omega_j}$, we arrive at

$$P_\uparrow(m|\{\xi\}) = \frac{1}{2m!}\left[\prod_{j=1}^m \int_{\mathcal{T}} \frac{dx_j}{2\pi\mathrm{i}\, x_j} \frac{1}{1+K\exp\Bigl(-\frac{h}{T} - \frac{2c}{T}\frac{1+x_j}{1-x_j}\Bigr)}\right]$$
$$\left[\prod_{\substack{j,k=1\\ j\ne k}}^m (x_j - x_k)\right]\frac{(1+\langle\sigma\rangle)\prod_{k=1}^m x_k + (1-\langle\sigma\rangle)\prod_{k=1}^m s_k}{\prod_{j,k=1}^m (x_j - s_k)}\,. \tag{51}$$

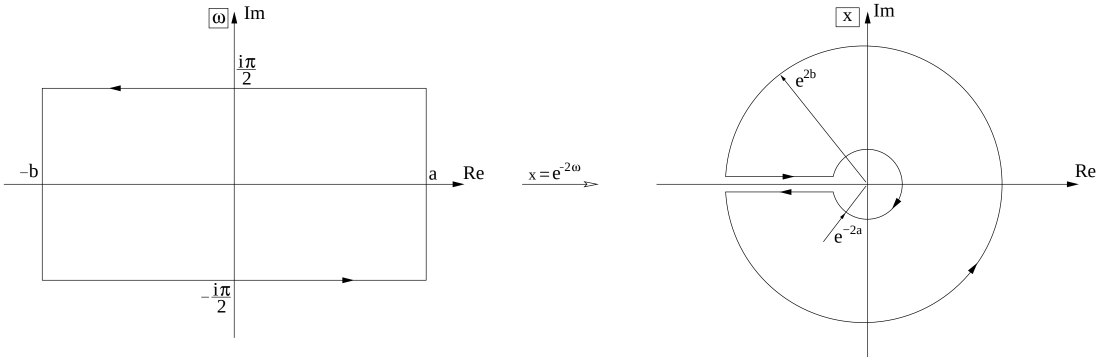

Figure 3: The transformation of the canonical contour under the map $\mathrm{e}^{-2\omega} \to x$.

Here we denoted the transformed contour by $\mathcal{T}$. It consists of two concentric circles about the origin. The left edge at, say, $-b$ of the original contour of integration is mapped to a large circle of radius e^{2b}, while the right edge at, say, a is mapped to a small circle of radius e^{-2a} (see figure 3). As the radius of the large circle is increased the auxiliary function on the contour tends to its limiting value $\mathfrak{a}_- = K\mathrm{e}^{-(h-2c)/T}$. On the small circle it approaches $\mathfrak{a}_+ = K\mathrm{e}^{-(h+2c)/T}$ if the radius goes to zero. Since the remaining factors in the integrand are rational in the x_j with only simple poles at $x_j = s_k$, $k = 1, \dots, m$, and at $x_j = 0$, we may calculate the integral by first fixing a large and a small circle, such that all poles of the rational part of the integrand, except the poles at zero, lie in the annulus between the circles, then replacing the auxiliary function by $\mathfrak{a}_-$ on the large circle and $\mathfrak{a}_+$ on the small circle and finally using the residue theorem. After carrying out n integration this way the right hand side of (51) takes the form

$$
\begin{aligned}
P_\uparrow(m|\{\xi\}) = {} & \frac{1}{2m!(1+\mathfrak{a}_-)^n} \left[\prod_{j=1}^{m-n} \int_{\mathcal{T}} \frac{dx_j}{2\pi \mathrm{i} x_j} \frac{1}{1 + K\exp\left(-\frac{h}{T} - \frac{2c}{T}\frac{1+x_j}{1-x_j}\right)} \right] \left[\prod_{\substack{j,k=1 \\ j\neq k}}^{m-n} (x_j - x_k) \right] \\
& \sum_{j_1=1}^{m} \sum_{\substack{j_2=1 \\ j_2\neq j_1}}^{m} \cdots \sum_{\substack{j_{n-1}=1 \\ j_{n-1}\neq j_1,\dots,j_{n-2}}}^{m} \left\{ \sum_{\substack{j_n=1 \\ j_n \neq j_1,\dots,j_{n-1}}}^{m} \frac{\prod_{k=1}^{m-n} \prod_{\ell=1}^{n} (s_{j_\ell} - x_k)}{\prod_{\substack{k=1 \\ k\neq j_1,\dots,j_n}}^{m} \prod_{\ell=1}^{n} (s_{j_\ell} - s_k)} \right. \\
& \qquad \frac{(1+\langle\sigma\rangle)\prod_{k=1}^{m-n} x_k + (1-\langle\sigma\rangle) \prod_{\substack{k=1 \\ k\neq j_1,\dots,j_n}}^{m} s_k}{\prod_{j=1}^{m-n} \prod_{\substack{k=1 \\ k\neq j_1,\dots,j_n}}^{m} (x_j - s_k)} \\
& \left. -n(1-\langle\sigma\rangle) \frac{\mathfrak{a}_+ - \mathfrak{a}_-}{1+\mathfrak{a}_+} \frac{\left[\prod_{k=1}^{m-n}\prod_{\ell=1}^{n-1}(s_{j_\ell} - x_k)\right]\left[\prod_{\ell=1}^{n-1} s_{j_\ell}\right]}{\prod_{\substack{k=1 \\ k\neq j_1,\dots,j_{n-1}}}^{m} \prod_{\ell=1}^{n-1}(s_{j_\ell} - s_k)} \frac{\prod_{k=1}^{m-n} x_k^2}{\prod_{j=1}^{m-n} \prod_{\substack{k=1 \\ k\neq j_1,\dots,j_{n-1}}}^{m} (x_j - s_k)} \right\}
\end{aligned}
\tag{52}
$$

which is most easily verified by induction over n.

For $n = m$ numerous trivial cancellations among the products occur, and we obtain

$$P_\uparrow(m|\{\xi\}) = \frac{1}{(1+\mathfrak{a}_-)^m}\left[1 - \frac{1-\langle\sigma\rangle}{2}\,\frac{\mathfrak{a}_+ - \mathfrak{a}_-}{1+\mathfrak{a}_+}\sum_{k=1}^m \prod_{\substack{\ell=1\\ \ell\neq k}}^m \frac{s_\ell}{s_\ell - s_k}\right]. \tag{53}$$

This further simplifies due to the identity

$$\sum_{k=1}^m \prod_{\substack{\ell=1\\ \ell\neq k}}^m \frac{s_\ell}{s_\ell - s_k} = 1 \tag{54}$$

which holds for any set of mutually distinct complex numbers s_k, $k = 1,\dots,m$. Inserting (54) and (45), (46) into (53) we end up with

$$P_\uparrow(m|\{\xi\}) = \frac{1+\langle\sigma\rangle}{2(1+\mathfrak{a}_-)^{m-1}}\,. \tag{55}$$

where $\langle\sigma\rangle$ and $\mathfrak{a}_-$ are explicitly shown in equations (47) and (42), (43). Quite remarkably, the right hand side of (55) is completely independent of the inhomogeneities $\{\xi\}$.

In order to express the emptiness formation probability in more physical terms let us calculate the free energy in the Ising limit from the integral (6). Proceeding as above we find

$$\begin{aligned} f(T,h) &= -\frac{h}{2} - T\ln(1+\mathfrak{a}_-) \\ &= -T\ln\left(\mathrm{ch}\left(\frac{h}{2T}\right) + \sqrt{\mathrm{sh}^2\left(\frac{h}{2T}\right) + \mathrm{e}^{\frac{4c}{T}}}\right). \end{aligned} \tag{56}$$

Comparison of (55) with (49) and (56) then yields

$$P_\uparrow(n+1) = \left(\tfrac{1}{2} + m(T,h)\right)\mathrm{e}^{\frac{nf(T,h)}{T} + \frac{nh}{2T}} \tag{57}$$

in accordance with [27] and with the general formula for the asymptotics of the emptiness formation probability at finite temperature [5].

5 Conclusions

Our aim in this article was to demonstrate the usefulness of finite temperature multiple integral representations for correlation functions, as obtained in [8–10], by showing that a number of known results for the correlation functions of the XX chain and of the one-dimensional Ising model can be reproduced by means of our formulae (7), (11) and (12). Usually the correlation functions of the XX and of the Ising chain are calculated from rather different, very special devices. The correlation functions of the XX chain are obtained by mapping the chain onto a model of free Fermions [24], while

the calculation of the correlation functions of the one-dimensional Ising model relies on the transfer matrix formalism with a spectral parameter independent 2×2 transfer matrix [2]. Here, for the first time, we have obtained correlation functions for the XX chain and for the Ising chain from a common device, namely the multiple integral representation, which interpolates for all values of Δ between the XX limit $\Delta = 0$ and the Ising limit $\Delta \to \infty$.

We would like to see this work in a line with [28], where a high-order high-temperature expansion for correlation functions of the isotropic chain was obtained starting from the multiple integrals, and with [7], where, again from the multiple integrals, some of the two- and three-site functions were computed numerically. We hope to report on further applications of the finite temperature integral formulae in the future.

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References

[1] E. Barouch and B. M. McCoy, *Statistical mechanics of the XY model. II. Spin correlation functions*, Phys. Rev. A **3** (1971), 786.

[2] R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London, 1982.

[3] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, and Y. Takeyama, *A recursion formula for the correlation functions of an inhomogeneous XXX model*, preprint, hep-th/0405044, 2004.

[4] ______, *Reduced qKZ equation and correlation functions of the XXZ model*, preprint, hep-th/0412191, 2004.

[5] H. E. Boos and V. E. Korepin, *Quantum spin chains and Riemann zeta function with odd arguments*, J. Phys. A **34** (2001), 5311.

[6] H. E. Boos, V. E. Korepin, and F. A. Smirnov, *Emptiness formation probability and quantum Knizhnik-Zamolodchikov equation*, Nucl. Phys. B **658** (2003), 417.

[7] M. Bortz and F. Göhmann, *Exact thermodynamic limit of short range correlation functions of the antiferromagnetic XXZ chain at finite temperatures*, preprint, cond-mat/0504370, 2005.

[8] F. Göhmann, A. Klümper, and A. Seel, *Emptiness formation probability at finite temperature for the isotropic Heisenberg chain*, preprint, cond-mat/0406611, to be published in Physica B, 2004.

[9] ———, *Integral representations for correlation functions of the XXZ chain at finite temperature*, J. Phys. A **37** (2004), 7625.

[10] ———, *Integral representation of the density matrix of the XXZ chain at finite temperature*, J. Phys. A **38** (2005), 1833.

[11] M. Inoue and M. Suzuki, *The ST-transformation approach to analytic solutions of quantum systems. II. Transfer-matrix and Pfaffian methods*, Prog. Theor. Phys. **79** (1988), 645.

[12] A. G. Izergin and V. E. Korepin, *The quantum inverse scattering method approach to correlation functions*, Comm. Math. Phys. **94** (1984), 67.

[13] M. Jimbo, K. Miki, T. Miwa, and A. Nakayashiki, *Correlation functions of the XXZ model for* $\Delta < -1$, Phys. Lett. A **168** (1992), 256.

[14] M. Jimbo and T. Miwa, *Quantum KZ equation with* $|q| = 1$ *and correlation functions of the XXZ model in the gapless regime*, J. Phys. A **29** (1996), 2923.

[15] G. Kato, M. Shiroishi, M. Takahashi, and K. Sakai, *Next-nearest-neighbour correlation functions of the spin-1/2 XXZ chain at the critical region*, J. Phys. A **36** (2003), L337.

[16] ———, *Third-neighbour and other four-point correlation functions of spin-1/2 XXZ chain*, J. Phys. A **37** (2004), 5097.

[17] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, *Correlation functions of the XXZ spin-*$\frac{1}{2}$ *Heisenberg chain at the free fermion point from their multiple integral representations*, Nucl. Phys. B **642** (2002), 433.

[18] ———, *Spin-spin correlation functions of the XXZ-*$\frac{1}{2}$ *Heisenberg chain in a magnetic field*, Nucl. Phys. B **641** (2002), 487.

[19] ———, *Dynamical correlation functions of the XXZ spin-1/2 chain*, preprint, hep-th/0407108, 2004.

[20] ———, *On the spin-spin correlation functions of the XXZ spin-*$\frac{1}{2}$ *infinite chain*, preprint, hep-th/0407223, 2004.

[21] N. Kitanine, J. M. Maillet, and V. Terras, *Correlation functions of the XXZ Heisenberg spin-*$\frac{1}{2}$ *chain in a magnetic field*, Nucl. Phys. B **567** (2000), 554.

[22] A. Klümper, *Free energy and correlation length of quantum chains related to restricted solid-on-solid lattice models*, Ann. Physik **1** (1992), 540.

[23] ———, *Thermodynamics of the anisotropic spin-1/2 Heisenberg chain and related quantum chains*, Z. Phys. B **91** (1993), 507.

[24] E. H. Lieb, T. Schultz, and D. Mattis, *Two soluble models of an antiferromagnetic chain*, Ann. Phys. (N.Y.) **16** (1961), 407.

[25] M. Shiroishi, M. Takahashi, and Y. Nishiyama, *Emptiness formation probability for the one-dimensional isotropic XY model*, J. Phys. Soc. Jpn. **70** (2001), 3535.

[26] M. Suzuki, *Transfer-matrix method and Monte Carlo simulation in quantum spin systems*, Phys. Rev. B **31** (1985), 2957.

[27] ______, *Asymptotics and thermodynamics of spin pattern formation probability in equilibrium spin systems*, Phys. Lett. A **301** (2002), 398.

[28] Z. Tsuboi and M. Shiroishi, *High temperature expansion of emptiness formation probability for isotropic Heisenberg chain*, preprint, cond-mat/0502569, 2005.